Allocating Systematic and Unsystematic Risks in a Regulatory Perspective

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Summary

1. AIMS OF THE PAPER

Typical formula for the required capital of a given entity at t. (Basel regulation)

$$RC_t = \max(VaR_t, k\frac{1}{60}\sum_{h=0}^{59} VaR_{t-h})$$

Potential drawbacks :

- Considers each entity separately, without any reference to the global system (bottom-up approach)
- Does not distinguish systematic and unsystematic sources of risk
- May imply procyclicity

In this paper we

- Introduce axioms that should be satisfied by the contributions of individual entities to a global risk (top-dow approach)
- Characterize the contributions which satisfy these axioms
- Compare with alternative approaches
- Decompose these contributions in a systematic and unsystematic part
- Discuss the use of these decompositions in a regulatory perspective.

2. THREE AXIOMS

2.1 Notations

A global $L\&P$ X is decomposed into :

$$X = \sum_{i=1}^{n} X_i$$

A global reserve (risk) $R(X)$, depending only on P the distribution of X, is defined, for instance a VaR, an Expected Shortfall or any other risk measure.

It has to be assigned to the different entities :

$$R(X) = \sum_{i=1}^{n} R(X, X_i)$$

$R(X, X_i)$ contribution of entity i to the global reserve (risk).

2.2 The axioms

A.1. Decentralization axiom

Axiom A1 : $R(X, X_i)$ depends on the joint distribution of (X, X_i) but not on the decomposition of $X - X_i$ into $\Sigma_{j \neq i} X_j$.

i.e. : Minimal confidentiality.

A.2. Additivity axiom

Axiom A2 : $R(X) = \sum_{i=1}^{n} R(X, X_i)$ for any decomposition of X into $\sum_{i=1}^{n} X_i$.

i.e. : the global reserve (risk) does not depend on the number of entities and on their respective sizes provided that X remains the same.

Avoid spurious advantages of merging or demerging in terms of global reserve, when the global $L\&P$ remains the same.

Implies that $X_i \to R(X, X_i)$ is additive.

A.3. Risk ordering axiom

We introduce a notion of directional stochastic dominance :

X_1^* stochastically dominates X_1 at order 2, with respect to X iff :

$$E[U(X_1^*, X - X_1^*)|X] \geq E[U(X_1, X - X_1)/X]$$

for any concave function U (denoted $X_1^* \geq_X X_1$)

Axiom A.3 : $R(X, X_1^*) \leq R(X, X_1)$ when $X_1^* \succeq_X X_1$

3. CHARACTERIZATIONS OF THE CONTRIBUTIONS

The contributions satisfying A1, A2, A3 have two mathematically equivalent characterizations :

$$i) \boxed{R(X, X_i) = \int E(X_i | X = x) \mu_P(dx)}$$

where μ_P is a measure on the range of X satisfying :

$$R(X) = \int x \mu_P(dx)$$

$$ii) \boxed{R(X, X_i) = \int E[X_i | X = q_\alpha(X)] \nu_P(d\alpha)}$$

where $q_\alpha(X)$ is the α-quantile of X and ν_P is a measure on $[0, 1]$ satisfying :

$$R(X) = \int q_\alpha(X) \nu_P(d\alpha)$$

ν_P : Allocation Distortion Measure (ADM)

Two remarks :

i) If $R(X)$ is a Distortion Risk Measure (DRM) we have :

$$R(X) = \int q_\alpha(X)H(d\alpha)$$

where H is a probability measure on $[0, 1]$ not depending on P, (for instance, if H is the point mass at $\alpha^* R(X)$ is $VaR(\alpha^*)$, if H is the uniform distribution on $[\alpha^*, 1]$ $R(X)$ is $ES(\alpha^*)$).
But ν_P does not have to be equal to H (can depend on P)

ii) If $(X_1, \ldots, X_n)$ is Gaussian all the contributions satisfying A_1, A_2, A_3 are the same, and equal to :

$$E(X_i) + \frac{Cov(X_i, X)}{V(X)}[R(X) - E(X)]$$

(expected individual $L\&P$ + hedging term, hedging term = beta $\times$ economic capital)

3. ALTERNATIVE APPROACHES

3.1 CoVaR (Adrian-Brunnermeier (2009))

CoVaR for entity *i* at level α defined by :

$$P[X_i < CoVaR_{i/s,\alpha}(X) | X = q_\alpha(X)] = \alpha$$

$$R(X, X_i) = CoVaR_{i/s,\alpha}(X)$$

Decentralization axiom satisfied

$\sum_{i=1}^{n} CoVaR_{i/s,\alpha}(X)$ is not a function of the distribution of X only and additivity axiom not satisfied

Contribution of *i* to the systematic risk defined by $CoVaR_{is,\alpha}(X) - q_\alpha(X_i)$

3.2 Shapley value (Tarashev et al. (2009))

Shapley value : a fair allocation of gains obtained by cooperation among several actors, based on a function $v(s)$ measuring the value of this cooperation for a coalition $S \subset \{1, \ldots, n\}$

Idea : use $v(S) = -R(\Sigma_{i \in S} X_i)$, where R is a risk measure (VaR or ES), to define the contributions

- Assume cooperation
- Does not satisfy the decentralization axiom.

3.3 Euler allocations (Tasche (2008))

When $R(X)$ is homogeneous of order 1

$R(\lambda X) = \lambda R(X)$ or, equivalently, defining $X(\wedge) = \sum_{i=1}^{n} \lambda_i X_i$,

$R(\sum_{i=1}^{n} \lambda_i X_i) = R[X(\wedge)] = R^*(\wedge)$, and therefore $R(X) = R^*(e)$,

we have $R^*(\lambda e) = \lambda R^*(e)$ and, using the Euler's identity :

$$R(X) = R^*(e) = \sum_{i=1}^{n} \frac{\partial R^*(e)}{\partial \lambda_i}$$

$$R(X, X_i) = \frac{\partial R^*(e)}{\partial \lambda_i}$$

Sensitivity interpretation : Marginal effect on the global reserve (risk) of a shock on the L&P of entity i

For instance if $R(X)$ is a DRM :

$$R(X) = \int q_\alpha(X) H(d\alpha)$$

or

$$R^*(e) = \int q^*_\alpha(e) H(d\alpha)$$

we get :

$$R(X) = \sum_{i=1}^{n} \int \frac{\partial q^*_\alpha(e)}{\partial \lambda_i} H(d\alpha)$$

and

$$\boxed{R(X, X_i) = \int \frac{\partial q^*_\alpha(e)}{\partial \lambda_i} H(d\alpha)}$$

> Lemma 1 : Let us consider the quantile $q_\alpha(\beta Z + Y)$, the sensitivity parameter
> $\frac{\partial q_\alpha(\beta Z + Y)}{\partial \beta}$ is equal to $E[Z|\beta Z + Y = q_\alpha(\beta Z + Y)]$

$\Longrightarrow$ Euler allocation $\boxed{R(X, X_i) = \int E[X_i / X = q_\alpha(X)] H(d\alpha)}$

Therefore : $R(X, X_i)$ satisfies the second characterization but :
$ADM = DM, (\nu_P = H)$
and ν_P does not depend on P.

Particular cases

- $R(X)$ is $VaR_{\tilde{\alpha}}$, i.e. H is the Point Mass at $\tilde{\alpha}$:

Euler allocation : $\boxed{R(X, X_i) = E[X_i / X = q_{\tilde{\alpha}}(X)]}$

- $R(X)$ is the Expected Shortfall $ES_{\tilde{\alpha}}$, i.e. H is the Uniform Distribution on $[\tilde{\alpha}, 1]$:

$$\begin{aligned} R(X, X_i) &= \frac{1}{1-\tilde{\alpha}} \int_{\tilde{\alpha}}^{1} \frac{q^*_{\alpha}(e)}{\partial \lambda_i} d\alpha \\ &= \frac{\partial ES^*_{\tilde{\alpha}}(e)}{\partial \lambda_i} \end{aligned}$$

Lemma 2 : Let us consider the Expected Shortfall $ES_\alpha(\beta Z + Y)$ (defined by $ES_\alpha(\beta Z + Y) = E[\beta Z + Y | \beta Z + Y > q_\alpha(\beta Z + Y)]$ the sensitivity parameter

$\frac{\partial ES_\alpha(\beta Z + Y)}{\partial \beta}$ is equal to $E[Z/\beta Z + Y > q_\alpha(\beta Z + Y)]$

Euler Allocation : $\boxed{R(X, X_i) = E[X_i / X > q_{\tilde{\alpha}}(X)]}$

However, in this Expected Shortfall case, where $R(X) = ES_{\tilde{\alpha}}$ there are many other allocations satisfying A1,A2,A3, i.e. of the form :

$$R(X, X_i) = \int E(X_i | X = x) \mu_P(dx)$$

$$\text{with} \quad ES_{\tilde{\alpha}} = \int x \mu_P(dx)$$

for instance if μ_P is the Point Mass at $ES_{\tilde{\alpha}}$:

$$R(X, X_i) = E(X_i | X = ES_{\tilde{\alpha}})$$

4.CONTRIBUTIONS TO SYSTEMATIC RISK

4.1 Linear factor model

$$X_i = \sum_{k=1}^{K} \beta_{ik} f_k + \gamma_i u_i = X_{s,i} + X_{u,i}$$

$f_1, \ldots, f_K$ systematic factors

$u_1, \ldots, u_n$ idiosynchratic terms (independent of the $f_k' s$)

$$X = \sum_{k=1}^{K} \beta_k f_k + \sum_{i=1}^{n} \gamma_i u_i$$

$$\text{with } \beta_k = \sum_{i=1}^{n} \beta_{ik}$$

$R(X)$: function of P

P : function of $\beta_1, \ldots, \beta_K, \gamma_1, \gamma_n, \theta$

θ : parameter characterizing the distribution of $(f_1, \ldots, f_k, u_1, \ldots, u_n)$

$$\begin{aligned} R(X, X_i) &= \int E|X_i/X = q_\alpha(X)]\nu_P(d\alpha) \\ &= R(X, X_{s,i}) + R(X, X_{u,i}) \\ &= \sum_{k=1}^{K} \beta_{ik} \int E[f_k/X = q_\alpha(x)]\nu_P(d\alpha) + \gamma_i \int E[u_i/X = q_\alpha(X)]\nu_P(d\alpha) \end{aligned}$$

$$\boxed{R(X, X_i) = R_s(X, X_i) + R_u(X, X_i)}$$

$$\begin{aligned} \text{with} \quad R_s(X, X_i) &= \sum_{k=1}^{K} \beta_{ik} R^f(X, f_k) \text{ systematic part} \\ R^f(X, f_k) &= \int E[f_k/X = q_\alpha(X)]\nu_P(d\alpha) \\ R_u(X, X_i) &= \gamma_i R_i^u(X, u_i) \text{ unsystematic part} \\ R^u(X, u_i) &= \int E[u_i/X = q_\alpha(X)]\nu_P(d\alpha) \end{aligned}$$

Finally :

$$R(X) = \sum_{i=1}^{n} R(X, X_i) = \sum_{i=1}^{n} R_s(X, X_i) + \sum_{i=1}^{n} R_u(X, X_i)$$

$$\boxed{R(X) = R_s(X) + R_u(x) \text{ (say)}}$$

Examples

• $R(X) = q_\alpha(X)$ (VaR Case)
(i.e. $R(X)$ is a DRM with H Point Mass at α)
and : $\nu_P = H$ (Euler allocation)

then :

$$R^f(X, f_k) = E[f_k/X = q_\alpha(X)]$$
$$R^u(X, u_i) = E[u_i/X = q_\alpha(X)]$$

•$R(X) = ES_\alpha(X)$ (Expected Shortfall case)
H uniform distribution on $[\alpha, 1]$, and $\nu_P = H$
replace = by >

Case of large number of entities ($n = +\infty$)

$$\lim_{n \to \infty} R(X, X_i) = \lim_{n \to \infty} R_s(X, X_i)$$

the unsystematic part disappears

(situation considered in Acharya et al (2010) and Brownlees Engle (2010))

4.2 Nonlinear factor model

$$X_i = g_i(f, u_i)$$

f, u_i independent

$$\begin{aligned} X_i &= E(X_i/f) + [E(X_i/u_i) - E(X_i)] \\ &\quad +[X_i - E(X_i/f) - E(X_i/u_i) + E(X_i)] \\ &= X_{s,i} + X_{u,i} + X_{u,s,i} \\ R(X, X_i) &= R_s(X, X_i) + R_u(X, X_i) + R_{s,u}(X, X_i) \end{aligned}$$

with $R_s(X, X_i) = \int E[X_{s,i}/X = q_\alpha(X)] d\nu_P(\alpha)$

and similar expressions for $R_u(X, X_i)$, $R_{s,u}(X, X_i)$ with

$$R(X) = \int q_\alpha(X) d\nu_P(\alpha)$$

5. REQUIRED CAPITAL

5.1 Link function

A "Natural" link function would be :

$$RC_{i,t} = \max[R(X_t, X_{it}), k_t \frac{1}{60} \sum_{h=0}^{59} R(X_{t-h}, X_{i,t-h})$$

but the two types of risks should be distinguished, for instance,

$$\begin{aligned} RC_{i,t} &= \max[R(X_t, X_{u,i,t}), k_{u,t} \frac{1}{60} \sum_{h=0}^{59} R(X_{t-h}, X_{u,i,t-h}) \\ &+ k_{s,t} \frac{1}{H} \sum_{h=0}^{H-1} R(X_{t-h}, X_{s,i,t-h}) \\ &\equiv RC^u_{i,t} + RC^s_{i,t} \end{aligned}$$

5.2 Change of global risk measure

Previous idea : transform a "Point In Time" (PIT) measure $R(X_t, X_{i,t})$, into a "Through The Cycle" (TTC) measure $RC_{i,t}$ depending on current and lagged values

Two step approach lacks coherency, for instance, additivity not satisfied

One step TTC approach ?

In the factor model framework the global measure $R(.)$ should depend not only on X but also on F.

$$R(F_t, X_t)$$

For instance,

$$R(X, F) = E(X) + E[(X - c(F))^+]$$

The basic results becomes :

$$R(X, F, X_i) = \int E(X_i / X = q_\alpha(X)) \nu_{P^*}(d\alpha)$$

P^* distribution of (X, F)

with $\int q_\alpha(X) \nu_{P^*}(d\alpha) = R(X, F)$

6. CONCLUSION

- "Allocation" or "contribution" problem different from Risk measurement problem
- Contributions are contingent to the level global risk (and possibly macroeconomic factors, e.g. position in the cycle)
- Axioms do not define a unique contribution
- The ADM approach can also be used to disentangle systematic and unsystematic components
- The ADM approach implies no restriction on the global risk measure.